The Mathematics of Sudoku

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1 Introduction

Sudoku is a puzzle presented on a square grid that is usually 9×9, but is sometimes 16×16 or other sizes. In this document, we will consider only the 9×9 case, although almost everything that is said can easily be extended to puzzles with different dimensions. Sudoku puzzles can be found in many daily newspapers, and there are thousands of references to it on the internet.

	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
a			4	8					
b		9		4	6			7	
c		5					6	1	4
d	2	1		6			5		
e	5	8		7		9		4	1
f			7			8		6	9
g	3	4	5					9	
h		6			3	7		2	
i						4	1		

Figure 1: An easy sudoku puzzle

The original grid has some of the squares filled with the digits from 1 to 9 and the goal is to complete the grid so that every row, column and 3×3 sub-grid (of which there are 9) contains each of the digits exactly once. Some initial configurations admit zero solutions and others admit multiple solutions, but these are usually considered to be invalid puzzles.

In figure 1 a (relatively easy) puzzle appears on the left. If you've never tried to solve a sudoku puzzle, it would be very informative to try to solve this one now, and see what strategies you can come up with before you read the rest of this article. It will probably take more time than you think, and you will get

much better with practice. And if you are a beginner, even if you're used to working the New York Times Sunday crossword in pen, use a pencil the first time you try sudoku!

Sudoku is interesting both as a logical exercise (What are good strategies for finding a solution?) and as a mathematical object (How many sudoku grids are there? How many are essentially different? What is the underlying mathematics and logic behind some solution techniques?). All of these aspects will be explored here.

We will first discuss various solution techniques and any related mathematics will be discussed along with the technique. At the end we'll include some information that is mostly mathematical and probably not of too much help in finding sudoku solutions.

In what follows, we will use the following terminology. There is a large literature on sudoku on the internet, and as far as possible, we will try to use the same terminology in this article as that which is commonly used on the internet.

- A "square" refers to one of the 81 boxes in the sudoku grid, each of which is to be filled eventually with a digit from 1 to 9.
- A "block" refers to a 3×3 sub-block of the main puzzle in which all of the numbers must appear exactly once in a solution. We will refer to a block by its columns and rows. Thus block $ghi456$ includes the squares $g4$, $g5$, $g6$, $h4$, $h5$, $h6$, $i4$, $i5$ and $i6$.
- A "candidate" is a number that could possibly go into a square in the grid. Many methods we will examine will eliminate candidates one after the other until there is a unique number that can go in a square.
- Sometimes an argument will apply equally well to a row, column or block, and to keep from having to write "row, column or block" over and over, we may refer to it as a "virtual line". A typical use of "virtual line" might be this: "If you know the values of 8 of the 9 entries in a virtual line, you can always deduce the value of the missing one." In the 9×9 sudoku puzzles there are 27 such virtual lines.
- Sometimes you would like to talk about all of the squares that cannot contain the same number as a given square since they are in the same row, column, or block. These are sometimes called the "buddies" of that square. For example, you might say something like, "If two buddies of a square have only two possible candidates, then you can eliminate those as candidates for the square."

2 Obvious Strategies

We will begin with a few strategies that are in a sense totally obvious, although searching for them in a puzzle may sometimes be difficult, since there are a lot of things to look for. Most puzzles have a difficulty rating, and almost all easy puzzles and many intermediate puzzles can be completely solved using only the techniques mentioned in this section. We'll begin with the most obvious observations and proceed to a few that are a bit more interesting. The methods are presented roughly in order of increasing difficulty for a human. For a computer, completely different approaches are possible and are often simpler.

2.1 Unique Missing Candidate

If eight of the nine elements in any row, column or block (or "virtual line") are already determined, the final element has to be the one that is missing. This technique is used a lot toward the end of a solution when most of the squares are already filled in. A similar obvious statement is this: If eight of the nine values are impossible in a given square, that square's value must be the ninth.

2.2 Naked Singles

	1	2	3	4	5	6	7	8	9
a	2 7 9	2 9	1	3	8	2 5 6 7 9	5 9	4	5 7 9
b	5	4	6	7 9	7 9	1	3 9	2	3 7 8 9
c	2 3 7 8 9	2 8 9	2 3 7 8	2 4 5 6 7 9	2 5 6 7 9	2 4 5 6 7 9	1 3 5 9	1 5 6 7 8	3 5 7 8 9
d	6	1 2 8	2 7 8	1 2 5 7	1 2 5 7	2 5 7 8	4	9	2 3 5 7
e	4	2	5	2 6 7 9	3	2 6 7 9	8	7	1
f	1 2 7 8	3	9	1 2 4 5 7	1 2 5 7	2 4 5 7 8	2 5	5 7	6
g	1 2 3 8 9	1 2 5 8 9	2 3 4 8	1 2 5 6 7 9	1 2 5 6 7 9	2 5 6 7 9	1 2 5 9	1 5 8	2 5 8 9
h	1 2 9	7	2	8	1 2 5 9	2 5 9	6	3	4
i	1 2 8 9	6	2 8	1 2 5 9	4	3	7	1 5 8	2 5 8 9

	1	2	3	4	5	6	7	8	9
a	2 7 9	9	1	3	8	2 5 6 7 9	5 9	4	5 7 9
b	5	4	6	7 9	7 9	1	3 9	2	3 7 8 9
c	2 3 7 8 9	8 9	3 7 8	2 4 5 6 7 9	2 5 6 7 9	2 4 5 6 7 9	1 3 5 9	1 5 6 8	3 5 7 8 9
d	6	1 8	7 8	1 2 5 7	1 2 5 7	2 5 7 8	4	9	2 3 5
e	4	2	5	6 9	3	6 9	8	7	1
f	1 7 8	3	9	1 2 4 5 7	1 2 5 7	2 4 5 7 8	2 5	5	6
g	1 3 8 9	1 5 8 9	3 4 8	1 2 5 6 7 9	1 2 5 6 7 9	2 5 6 7 9	1 2 5 9	1 5 8	2 5 8 9
h	1 9	7	2	8	1 5 9	5 9	6	3	4
i	1 8 9	6	8	1 2 5 9	4	3	7	1 5 8	2 5 8 9

Figure 2: Candidate Elimination and Naked Singles

For any given sudoku position, imagine listing all the candidates from 1 to 9 in each unfilled square. Then, for every square whose value v is determined, cross off every instance of v as a possible candidate in the row, column and block (or, in all three of the virtual lines) to which that square belongs. The remaining values in each square represent possible values that could possibly be inserted there. If, after such an elimination of the impossible candidates, only a single possible value remains, that situation is referred to as a "naked single" and that one remaining value can be assigned to the square.

In the example on the left side of figure 2 we see a sudoku puzzle where the larger numbers in the squares represent values that are already filled in. Squares whose values are not yet determined are filled with a list of possible candidates, where the values in the completed squares have been used to eliminate some values. After these obvious candidate eliminations have occurred, we can see that the puzzle contains three naked singles at $e2$ and $h3$ (where a 2 must be inserted), and at $e8$ (where a 7 must be inserted).

Notice that once you have filled in these values, other naked singles will appear. For example, as soon as the 2 is inserted at $h3$, you can eliminate the 2's as candidates in its row, column and block, and when this is done, $i3$ will become a naked single that must be filled with 8. The position on the right side of figure 2 shows the result of the previous puzzle after the three squares mentioned in the previous paragraph have been filled and the obvious candidates have been eliminated from the unfilled squares.

2.3 Hidden Singles

Sometimes there are cells that do, in fact, have only one possible value based on the situation, but a simple elimination of candidates in that square's row, column and block do not make it obvious. If you reexamine the situation on the left side of figure 2, there is a hidden single in square $g2$ whose value must be 5. The two 5's in $b1$ and $e3$ require that the 5 that must appear in the lower-left block ($ghi123$) must occur in column 2. But there is only one available square in that block's column 2 that is not yet filled. Thus 5 can be placed in square $g2$. The 5 in square $g2$ is "hidden" in the sense that without further examination, it appears that the values 1, 2, 5, 8 and 9 are all possible candidates.

An easy way to find hidden singles is to look in every virtual line for a candidate that appears in only one of the squares. If that occurs, you've found a hidden single.

The example above is for a hidden single in a block. The same thing can occur in any virtual line. Using the same example in figure 2, there is a hidden single in square $d9$ where a 3 must be placed. A 3 must appear somewhere in row d, but 3's in the two leftmost blocks containing row d already contain a 3 so the 3 must go in $d7$, $d8$ or $d9$. Since squares $d7$ and $d8$ are already filled, $d9$ contains a hidden single.

The application of any of the techniques in this section immediately assigns a value to a square. Most puzzles that are ranked "easy" and many that are ranked "intermediate" can be completely solved using only these methods.

The remainder of the methods that we will consider do not automatically allow you to fill in a square. What they do is to eliminate certain candidates from certain squares. Obviously, once all the candidates but one have been eliminated, then the value to be placed in that square is completely determined.

3 Locked Candidates

Sometimes you can find a block where the only possible positions for a candidate are in one row (or column) within that block. Since the block must contain the candidate, the candidate must appear in that row (or column). But that means that you can eliminate the candidate as a possibility in the intersection of that row (or column) with other blocks.

A similar situation can occur when a number that must go into a row or column can occur only within one of the blocks that intersect that row or column. Thus the candidate must lie on the intersection of the row/column and block and hence cannot be a candidate in any of the other squares that make up the block.

All of these situations are illustrated in figure 3. The block $def789$ must contain a 2, and the only places this can occur are in squares $f7$ and $f8$: both in row f. Therefore 2 cannot be a candidate in any other squares in row f, including square $f5$ (so $f5$ must contain a 3). Similarly, the 2 in block $ghi456$ must lie in column 4 so 2 cannot be a candidate in any other squares of that column, including $d4$.

Finally, the 5 that must occur in column 9 has to fall within the block $def789$ so 5 cannot be a candidate in any of the other squares $d7$, including $f7$ and $f8$.

	1	2	3	4	5	6	7	8	9
a	1	4 5	8	6	7	2	4 5	9	3
b	7 5 3	7 5 3	9	8	1	4	6	7 5	2
c	4 7 2	6	7 2	9	5	3	8	4 7	1
d	4 5 9 3	4 5 3	6	4 5 2	2 3	7	1	8	5 9
e	4 5	2	1	4 5	9	8	7	3	6
f	4 5 7 9 3	8	7 5 3	1	2 3	6	4 5 2	4 5 2	5 9
g	5 2	1	4	3	8	9	5 2	6	7
h	6	7 5 3	7 2 5 3	7 2	4	1	9	5 2	8
i	8	9	7 2	7 2	6	5	3	1	4

Figure 3: Locked Candidates

4 Naked and Hidden Pairs, Triplets, Quads, . . .

These are similar to naked singles, discussed in section 2.2, except that instead of having only one candidate in a cell, you have the same two candidates in two cells (or, in the case of naked triplets, the same three candidates in three cells, et cetera).

The naked pair, triplet or quad can be in the same virtual line, and when it occurs, those values must use up all the squares. Thus those candidates are eliminated from any other square in that virtual line.

Figure 4 shows how a naked pair can be used. In squares $a2$ and $a8$ the only candidates that appear are a 2 and a 7. That means that those squares must be filled with those numbers, in some order. But that means that the 2 and 7 cannot be in any of the other squares in that row, so 2 can be eliminated as a candidate in $a3$ and both 2 and 7 can be eliminated as candidates in $a9$.

	1	2	3	4	5	6	7	8	9
a	1	7 2	4 2 3	5	4 3 6	8	9	7 2	4 7 2 3 6

Figure 4: A Naked Pair

In the case of naked pairs, both squares must have exactly the same two candidates, but in the case of naked triplets, quads, et cetera, the only requirement is that the three values be the *only* values appearing in those squares in some virtual line. For example, if three entries in a row admit the following sets of candidates: $\{1, 3\}$, $\{3, 7\}$ and $\{1, 7\}$ then it is impossible for a 1, 3 or 7 to appear in any other square of that row.

The idea is fairly simple and is illustrated in figure 5. In row a squares $a2$, $a8$ and $a9$ contain the naked triple consisting of the numbers 1, 3 and 7. Thus those numbers must appear in those squares in some

	1	2	3	4	5	6	7	8	9
a	5	1 7	2	1 4 8	1 3 4 8	9	6	3 7	1 3

Figure 5: A Naked Triple

order. For that reason, the candidates 1 and 3 can be eliminated from squares $a4$ and $a5$.

i	6 9	5	6 7 9	2 7	4	3	2 7 9	1	8

Figure 6: A Hidden Triple

Hidden pairs, triples and quads are related to naked pairs, triples and quads in much the same way that hidden singles are related to naked singles. In figure 6 consider row i. The only squares in row i in which the values 1, 4 and 8 appear are in squares $i1$, $i5$ and $i6$. Therefore we can eliminate candidates 2 and 6 from square $i1$ and candidate 3 from $i5$.

Remember, of course, that these hidden sets can appear in any virtual line: a row, column, or block. There is also no reason that there could not be a naked or hidden quintet, sextet, and so on, especially for versions of sudoku that are larger than 9×9.

5 X-Wings and Swordfish

An x-wing configuration occurs when the same candidate occurs exactly twice in two rows and in the same columns of those two rows. (Or similarly, if you exchange the words “row” and “column” in the previous sentence.) In the configuration on the left in figure 7 the candidate 3 occurs twice in rows c and h and in those two rows, it appears in columns 2 and 7. It does not matter that the candidate 3 occurs in other places in the puzzle.

The squares where the x-wing candidate (3, in this case) can go form a rectangle, so a pair of opposite corners of that rectangle must contain them. In the example, this means that the 3’s are either in $c2$ and $h7$ or they are in $c7$ and $h2$. Perhaps the fact that connecting the possible pairs would form an ‘X’, like the X-wing fighters in Star Wars gives this strategy its name.

In any case, since the two corners contain the candidate, no other squares in the columns or rows that contain the corners of the rectangle can contain that candidate. In the example, we can thus conclude that 3 cannot be a candidate in squares $a7$, $f7$ or $i2$.

A swordfish is just like an x-wing except that there must be three rows/columns with the three candidates appearing in at most three columns/rows. As was the case with naked and hidden triples, for a swordfish there is no requirement that the candidate to be in all three positions. The reasoning is similar to that used for the x-wing, however: once you find a swordfish configuration, the candidate cannot appear in any other squares of the three columns and rows.

A swordfish configuration appears on the right in figure 7. In this case, the candidate is 7, and the columns that form the swordfish are 2, 5 and 8. The value 7 appears only in rows a, f and i. One 7 must appear

	1	2	3	4	5	6	7	8	9
a	1 2 6	4	8	7	9	3 6	1 2 3 5	1 2 3 5	2 3
b	1 6 9	5	1 3	8	2	3 6	7	1 3 4	3 4 9
c	2 9	2 3 9	7	5	4	1	2 3 9	6	8
d	3	8	5	2	1	9	4	7	6
e	7	6	2	3	5	4	8	9	1
f	4	1	9	6	7	8	2 3	2 3	5
g	8	7	6	4	3	5	1 2 9	1 2	2 9
h	1 5 9	3 9	4	1 9	6	2	3 5	8	7
i	1 2 5 9	2 3 9	1 3	1 9	8	7	6	3 4 5	3 4

	1	2	3	4	5	6	7	8	9
a	1 2 7 8	1 3 7 8	9	5 8	3 7	6	1 3 5	4	1 2 3 5
b	2 4 8	3 8	3 4 8	5 8 9	1	8 9	7	6	2 3 5
c	1 7	6	5	3 7	4	2	1 3	9	8
d	3	9	6	1 4 7	8	5	1 4	2	1 7
e	1 4 7	5	2	1 3 4 7	6	1 3 4 7	9	8	1 3 7
f	1 4 7 8	1 7 8	1 4 8	2	9	1 3 4 7	1 3 4 5	1 5 7	6
g	6	4	1 8	1 7 8 9	5	1 7 8 9	2	3	1 7 9
h	5 9	1 3 8	7	1 3 4 8 9	2	1 3 4 8 9	6	1 5	1 4 5 9
i	5 9	2	1 3	6	3 7	1 3 4 7 9	8	1 5 7	1 4 5 7 9

Figure 7: X-Wing (left) and Swordfish

in each of these rows and and each of the columns, so no other squares in those rows and columns can contain a 7. Thus the candidate 7 can be eliminated from $a1$, $f1$, $f6$, $i6$ and $i9$.

Of course there is nothing special about a 3×3 configuration; "super-swordfish" with 4, 5, or 6 candidates might be possible, but they are rare but not particularly difficult to spot. The "super-swordfish" with 4 rows and 4 columns is sometimes called a "jellyfish". If you are playing on a standard 9×9 grid, the most complex situation you would need to look for would be a jellyfish, since if there were a 5×5 super-swordfish, there would have to be in addition a 4×4 or smaller swordfish in the remaining rows or columns. It's too bad that there's no real need for the 5×5 super-swordfish, since in the web "literature", it's called a "squirmbag".

6 The XY-Wing

The basic idea of the xy-wing is this: Sometimes a square has two candidates. If we assume that the first is used, then that forces a certain conclusion. If, by assuming that the second is true, the same conclusion is forced, then that conclusion must be true since no matter how the initial choice is made, the conclusion must follow.

In the configuration in figure 8, suppose that there are two possible candidates in squares $b2$, $b5$ and $e2$, as shown. Consider the contents of square $b2$. If X is there, then there must be a Z in $e2$ and therefore Z cannot be a candidate in $e5$. But the other possibility is that $b2$ is a Y. In this case, $b5$ must be Z and again, $e5$ cannot be Z. Thus, in a configuration like this, you can eliminate Z as a candidate in square $e5$.

In a similar way, consider the configuration on the left in figure 9. If either X or Y is true, the three squares indicated by asterisks cannot have Z as a candidate. In a similar way, by examining the configuration on the right in the same figure, Z is eliminated as a candidate in two more squares indicated by asterisks.

	1	2	3	4	5	6
a						
b		XY			YZ	
c						
d						
e		XZ			*	
f						

Figure 8: XY-Wing

	1	2	3	4	5	6
a		XY			XZ	
b						
c	YZ			*	*	*

	1	2	3	4	5	6
a	*	XY	*		XZ	
b						
c	YZ					

Figure 9: XY-Wing

Obviously, the two configurations in figure 9 can be combined to make figure 10 where Z can be eliminated as a candidate in any of the squares marked with an asterisk.

An example of an xy-wing in an actual puzzle appears in figure 11. Notice that in squares $d8$ and $f7$ (both in the same block, $def789$) and in square $d1$ we have candidates $\{8, 9\}$, $\{3, 9\}$ and $\{8, 3\}$, respectively. Because of this, we can eliminate 3 as a candidate from squares $d7$, $f1$ and $f2$.

7 Coloring and Multi-Coloring

Coloring and multi-coloring are techniques that infer colors based on logical chains of deduction. The coloring method, especially, is simple enough that it can be done by hand.

7.1 Simple Coloring

Consider the example in figure 12 where we consider a few squares that contain the candidate 1. Let's assume for now that these are the only possible locations for 1 in the puzzle. Certain virtual lines contain exactly two places where the candidate 1 can go: row b, row i, column 3 and block $def123$. In each of these virtual lines, exactly one of the possible squares can contain a 1 and once it is selected, the other cannot.

But this creates a sort of "chain" if $f1$ contains 1, then $e3$ must not, and since $e3$ must not, $b3$ must, so $b6$ must not, $i6$ must, and $i9$ must not. If, on the other hand, $f1$ does not contain a one, the same series of virtual line interactions will force an alternating set of conclusions and every square in the chain will be forced to have the opposite value. In the figure we've marked the squares with $+$ and $-$ according to the assumption that $f1$ does contain a 1, but of course it may be the case that $f1$ does not contain 1, and all the $+$ and $-$ signs would be interchanged. Rather than using the $+$ and $-$ characters that imply presence or absence of a value it is easier simply to imagine coloring each square in the chain black or white, and

	1	2	3	4	5	6
a	*	XY	*		XZ	
b						
c	YZ			*	*	*

Figure 10: Combined XY-Wing

	1	2	3	4	5	6	7	8	9
a	2	8	9	4	7 5	1	7 5	6	3
b	7	6	4 5	3	2	9	8	1 5	1 4
c	1	4 5	3	8	6	7 5	4 9	7 5 9	2
d	3 8	1	2	7	4 5 8 9	4 5 8	4 3 9	8 9	6
e	9	4 7	6	2	4 8	3	1	7 8	5
f	3 5 8	3 5 7	4 5 7	1	8 9	6	3 9	2	4 7
g	6	3 5 7	5 7 8	9	3 4 8	4 7 8	2	1 5 7	1 7
h	3 5	9	1	6	3 7	2	7 5	4	8
i	4	2	7 8	5	1	7 8	6	3	9

Figure 11: XY-Wing Example

either all the black squares have a 1 and all the white squares do not, or the opposite.

Suppose now that for some candidate you've discovered such a chain[1] and have colored it in this alternating manner.

It may be that there are additional squares where the candidate could possibly occur that do not happen to lie in the colored chain. In the example above, suppose square $f1$ is colored black and so square $i9$ must be colored white. Consider the square $f9$ that lies at the intersection of $f1$'s row and $i9$'s column. Since $f1$ and $i9$ have opposite colors, exactly one of them *will* contain a 1, and therefore it is impossible for the square $f9$ to contain a 1, so 1 can be eliminated as a possible candidate in that square.

There's nothing special about a row-column intersection. Any time two oppositely-colored squares "intersect" via virtual lines or any sort in another square, the candidate can be eliminated as a possibility in that square.

This is probably easier to see with the concrete example displayed on the left in figure 13 where we consider the interactions between squares with 1 as a possible candidate. In row d, $d1$ and $d5$ are the only occurrences of candidate 1, so we color $d5$ black and $d1$ white. But $d1$ and $f3$ are the only possibilities for 1 in block $def123$, so since $d1$ is white, $f3$ is black. By similar reasoning, since $f3$ is black, $g3$ and

[1]For astute readers, it may not really be a chain, but it could be a tree, or even have loops, as long as the black/white alternation is preserved.

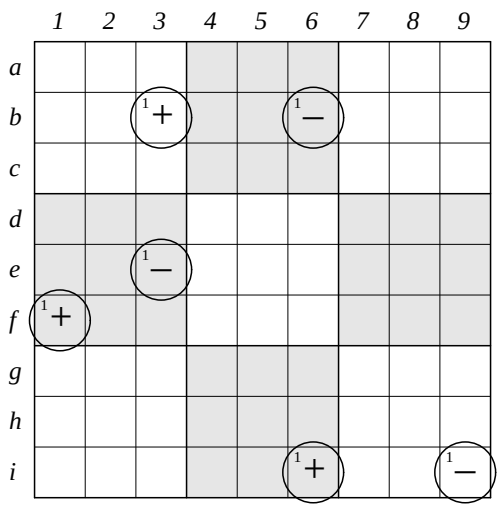

Figure 12: Simple Coloring

$f8$ are white. Since $f8$ is white, $e7$ is black, and since $e7$ is black, $c7$ is white. That's a pretty complicated chain, but here's what we've got: black: $\{d5, f3, e7\}$ white: $\{d1, g3, f8, c7\}$. A grid that displays just the colored squares appears on the right in figure 13.

Square $c5$ is at the intersection of $c7$'s row and $d5$'s column, but $c7$ is white and $d5$ is black, so 1 cannot be a candidate in square $c5$. Similarly, square $g5$ is in the same row as $g3$ and same column as $d5$ which are white and black, respectively, so 1 also cannot be a candidate in $g5$.

7.2 Multi-Coloring

Sometimes a position can be colored for a particular candidate and multiple coloring chains exist, but none of them are usable to eliminate that candidate from other squares. If there are multiple chains, it is worth looking for a multi-coloring situation.

Consider the puzzle in figure 14. Assume that in the parts of the puzzle that are not shown there are no other places that the candidate 1 can occur. When this diagram is colored, there are two coloring chains. Instead of using words like "black" and "white" we will used letters, like A, B, a and b where the A and a represent opposite colors, as do the B and b, and so on. In figure 14 rows a and c and in column 3 there are only two possible locations for candidate 1.

When this grid is colored, it will look something like this: squares $c1$ and $f3$ have color A and square $c3$ has color a. Square $a2$ has color b and square $a5$ has color B. (Note that the colors assigned are arbitrary. All that matters is that squares $c1$ and $f3$ have the same color that is the opposite of $c3$ and that $a2$ and $a5$ have opposite colors that are different from the other assigned colors. Note that none of the other squares with 1 as a candidate can be colored, since all are in virtual lines with more than two squares that potentially could contain the candidate 1.

If we consider the color "a" as standing for the sentence: "Every square containing the color a contains a

	1	2	3	4	5	6	7	8	9
a	8	6	7	1 2	1 2 3 5	9	4	1 3	2 3 5
b	4	1 3	3	2 6 7	2 5 6 7	8	5 7	9	2 5 6 7
c	5	2	9	1 4 6 7	1 3 6 7	1 4 7	1 3 7 8	1 3 7 8	3 6 7
d	1 7	4	2	5	1 7	6	3 8	3 8	9
e	9	3	8	1 2 7	4	1 2 7	1 5 7	6	5 7
f	6	5 7	1 5	8	9	3	2	1 7	4
g	1 2 3 7	9	1 6	1 2 4 6 7	1 2 6 7	1 2 4 7	3 7	5	8
h	2 7	8	5 6	3	2 6 7	2 5 7	9	4	1
i	1 3 7	5 7	4	9	8	1 5 7	6	2	3 7

	1	2	3	4	5	6	7	8	9
a									
b									
c							W		
d	W				B				
e							B		
f			B					W	
g			W						
h									
i									

Figure 13: Simple Coloring

	1	2	3	4	5	6
a	1 A	1			1	1
b						
c	1 a		1 A			
d						
e		1 b			1 B	
f		1	1 a		1	1

Figure 14: Multi-coloring

1," and so on, then we can write little logical expressions indicating the relationships among the various colors when they are interpreted as sentences. The obvious ones are of the form: "$\mathsf{a} = \neg\ \mathsf{A}$" or "$\mathsf{A} = \neg\ \mathsf{a}$" (where the logical symbol "$\neg$" means "not"). In other words, if a is true then A is not, and vice-versa.

Although the values of non-opposite colors do not necessarily have anything to do with each other, in figure 14, the pair a and b, for example are linked, since they occur in the same block. If a is true, then b cannot be, and vice-versa, but it may be true that both a and b are false. We will express this relationship as "a!b" and read it as "a excludes b". Obviously, if a!b then b!a[2]. Also, it's obvious in the configuration in figure 14 that b!A.

Another way to think of a!b is as "If a is true then so is B." Similarly, it means "If b is true then so is A." If a!b then at least one of a or b must be false. That means that at least one of A or B must be true. That means that any square at the generalized intersection of squares colored A and B must not allow the

[2]If you examine the truth table for a!b you will find that it is equivalent to the "nand" ("A nand B" is the same as "not(A and B)") logical operator that's heavily used in computer hardware logic designs.

candidate since one of the two squares colored A or B *must* contain the candidate. In figure 14, this means that 1 cannot be a candidate in square $f5$.

To condense all of the above into a single statement, we know that if a!b for some candidate then any square at the generalized intersection of A and B cannot contain that candidate.

	1	2	3	4	5	6	7	8	9
a	4	8	$_9$	$_9$	6	$_9$	2	7	5
b	2	5			7		9		6
c	$_9$	7	6		$_9$	$_9$	4	3	
d	5	2	$_9$	8	4	$_9$	6	$_9$	3
e	$_9$	6	8	3	5	2	7	4	$_9$
f	3	$_9$	$_9$	$_9$	1	6	8	5	2
g	8	3	5		$_9$	$_9$	1	6	7
h	6	1	$_9$		8		3	$_9$	4
i	7	$_9$	$_9$	6	3	1	5	$_9$	$_9$

	1	2	3	4	5	6	7	8	9
a			A_9	E_9		$_9$			
b									
c	a_9				b_9	$_9$			
d			$_9$			$_9$		A_9	
e	A_9								a_9
f		D_9	$_9$	e_9					
g					B_9	b_9			
h			C_9					c_9	
i		d_9	$_9$					$_9$	A_9

Figure 15: Multicoloring Example: Coloring on Right

But much more can be done. In complex situations, there may be many independent color chains with colors A and a, B and b, C and c, and so on. When that occurs, we need to look for consequences of the following inference:

If a!b and B!c then a!c.

It's not hard to see why: If a is true, b is not, so B is true, and the second exclusion implies that c is not. The reasoning is trivially reversed to show that if c is true then a is not, so we obtain a!c.

Thus to do multi-coloring for a particular candidate, proceed as follows:

- Construct all possible color chains for the diagram.
- Find all exclusionary relationships from pairs of colors that share the same generalized lines.
- Take the collection of relationships and complete it to its transitive closure using the idea that if (a!b and B!c) then a!c.
- For every exclusionary pair in the transitive closure, find generalized intersections of squares colored with colors opposite to those in the pair, and eliminate the candidate as a possibilty from all of them.

Let's look at a very complex multi-coloring application. See figure 15 where only the presence of squares that admit the candidate 9 are marked (all, of course, might admit other candidates). On the left is the complete grid and on the right is a simplified version where only the squares admitting candidate 9 are

shown, and all of the color chains are displayed. It is an excellent exercise to look at the diagram on the right to make certain that you understand exactly how all the color chains are constructed.

The next step in the application of multi-color is to find all the exclusionary pairs, and the initial list is the following. Note that the "!" operation is commutative, so if you think a!b should be in the list and it is not, be sure to look for b!a as well.

A!E	a!b	D!e	A!d	A!C
A!c	b!E	A!D	C!d	

From these initial exclusions, a number of others can be deduced. For example, from a!b and A!d we can conclude that b!d. Note that to make this implication, we are implicitly using the fact that a!b and b!a are equivalent.

In fact, if we make all such deductions, and then all deductions from those, and so on, there are ten additional exclusions that we find:

b!d	b!C	b!c	b!D	C!e
A!A	A!b	b!e	b!b	A!e

For most of them, we need to look for generalized intersections of the opposites of the exclusionary values. For example, since A!e and there is an a in $c1$ and a E in $a4$, then 9 cannot be a candidate in $c6$. Also, since we've got A!A and b!b we can conclude that a and B are true. To make it easier to check the consequences of these exclusionary implications, figure 16 shows the complete solution to the puzzle in figure 15.

	1	2	3	4	5	6	7	8	9
a	4	8	3	1	6	9	2	7	5
b	2	5	1	4	7	3	9	8	6
c	9	7	6	5	2	8	4	3	1
d	5	2	9	8	4	7	6	1	3
e	1	6	8	3	5	2	7	4	9
f	3	4	7	9	1	6	8	5	2
g	8	3	5	2	9	4	1	6	7
h	6	1	2	7	8	5	3	9	4
i	7	9	4	6	3	1	5	2	8

Figure 16: Solution to Multicoloring Example

8 Unique Solution Constraints

If you know that the puzzle has a unique solution, which any reasonable puzzle should, sometimes that information can eliminate some candidates. For example, let's examine the example in figure 17.

In row c, columns 4 and 6, the only possible candidates are 1 and 2. But in row g, columns 4 and 6, the candidates are 1, 2 and 8. We claim that 8 must appear in $g4$ or $g6$. If it does not, then the four corners of

	1	2	3	4	5	6	7	8	9
a	6	3 8	1	7	5	9	4	3 8	2
b	2	9	5	4 3 6	8	4 3	3 6	1	7
c	4	3 8	7	1 2	3 6	1 2	6 8 9	5	3 9
d	5	6	4 8	4 3 8 9	1	4 3 8	2	7	3 9
e	1	2	3 9	5	7	6	3 9	4	8
f	3 9	7	4 8	4 8 9	2	4 3 8	1	6	5
g	3 9	5	2 6 9	1 2 8	3 6	1 2 8	7	3 8 9	4
h	8	4	3 6	3 6	9	7	5	2	1
i	7	1	2 9	2 8	4	5	3 8 9	3 8 9	6

Figure 17: Uniqueness Constraint

the square $c4$, $c6$, $g4$ and $g6$ will all have exactly the same two candidates, 1 and 2, so we could assign the value 1 to either pair of opposite corners, and both must yield valid solutions. If there is a unique solution, this cannot occur, so one of $g4$ or $g6$ must contain the value 8. But if that's the case, square $i4$ cannot be 8, so the candidate 8 can be eliminated from square $i4$. In addition, since either $g4$ or $g6$ must be 8, $g8$ cannot be 8 since it is in the same row as the other two.

In the same figure, a similar situation appears in another place. See if you can find it. Hint: it column-oriented instead of row-oriented.

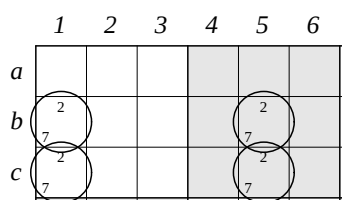

Figure 18: An illegal block

Let's go back and see exactly what is going on, and from that, we'll be able to find a number of techniques that are based on the same general idea. Figure 18 shows a basic illegal block. Anything at all can occur in the squares that are not circled, but note that an assignment of a 2 or a 7 to any of the circled squares forces the values of the others in an alternating pattern. But any of the squares can be assigned a 2 or a 7 and the resulting pattern will be legal, and this means there are two valid solutions to the puzzle.

This means that if some assignment causes an illegal block to be formed, that assignment is impossible, and we can use that fact to eliminate certain possibilities, as we did in the example in figure 17. Note that the four corners must not only form a rectangle, but they must be arranged so that two pairs of adjacent corners must lie within the same blocks. If the four corners lie in four different blocks, then constraints

from those different blocks can force the values one way or the other.

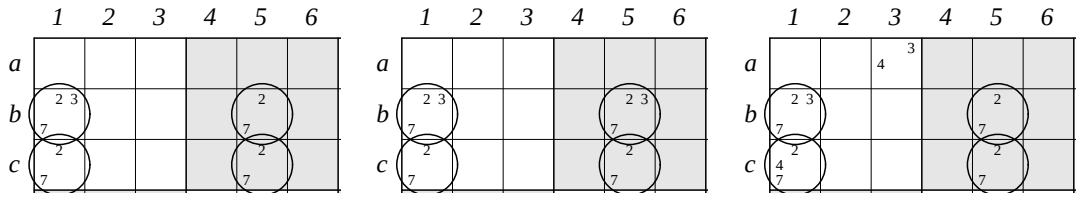

Figure 19: Uniqueness Considerations

Now let's examine some variations of this theme. In the rest of the examples in this section, we'll assume that empty squares can be filled with any valid puzzle entries: empty or determined. In figure 19 on the left we see something that is almost the same as what we saw in figure 18 and the only thing that makes it legal is the presence of the possibility of a 3 in square $b1$. If it is not a 3, then we would have the illegal block, so there must be a 3 in square $b1$. Note that if, in the figure, square $b1$ had contained the possibilities $2, 3, 4,$ and 7, at least the two possibilities 2 and 7 could still be eliminated as possibilities, so only a 3 or a 4 could be entered in that square.

The example in the middle of figure 19 is similar to the original example in this section except that the additional number occurs in two different blocks instead of one. As before, at least one of those squares must contain the number (3 in this case), so the value 3 can be eliminated from any of the other squares in that row (row b, in this case), but not in either of the blocks, since the one that is forced to be 3 might be in the other block.

The example on the right in figure 19 illustrates another sort of deduction that could be made. We know that at least one of $b1$ and $c1$ must contain a number other than a 2 or a 7, but we don't know which one. If we think of the combination of the two squares as a sort of unit, we *do* know that this unit will contain either a 3 or a 4. This two-square unit, together with square $a3$ (which has 3 and 4 as its unique possibilities) means that no other square in the block $abc123$ can contain a 3 or a 4. If the 34 square had been in $a1$ we could in addition eliminate 3 and 4 as candidates from any of the other squares in column 1 outside the first block.

Note that we can have both a 3 and a 4 in either or both squares $b1$ and $c1$ in this example on the right. As long as both occur, the argument holds. Also note that if the 3's and 4's appeared in row b and the entries in row c were both 27, and the 34 square were in row b we could eliminate any more 3's and 4's in that row.

9 Forcing Chains

This method is almost like guessing, but it is a form of guessing that is not too hard for a human to do. There are various types of forcing chains, but the easiest to understand works only with cells that contain two candidates.

The idea is this: for each of the two-candidate cells, tentatively set the value of that cell to the first value and see if that forces any other two-candidate cells to take on a value. If so, find additional two-candidate cells whose values are forced and so on until there are no more forcing moves. Then repeat the same

	1	2	3	4	5	6	7	8	9
a	1 2 6	4	8	7	9	3 6	1 2 3 5	1 2 3 5	2 3
b	1 6 9	5	1 3	8	2	3 6	7	1 3 4	3 4 9
c	2 9	2 3 9	7	5	4	1	2 3 9	6	8
d	3	8	5	2	1	9	4	7	6
e	7	6	2	3	5	4	8	9	1
f	4	1	9	6	7	8	2 3	2 3	5
g	8	7	6	4	3	5	1 2 9	1 2	2 9
h	1 5 9	3 9	4	1 9	6	2	3 5	8	7
i	1 2 5 9	2 3 9	1 3	1 9	8	7	6	3 4 5	3 4

Figure 20: Forcing Chains

operation assuming that the original cell had the other value.

If, after making all possible forced moves with one assumption and with the other, there exists a cell that is forced to the same value, no matter what, then that must be the value for that cell.

As an example, consider the example in figure 20, and let's begin with cell $b3$ which can contain either a 1 or a 3. If $b3 = 1$, then $i3 = 3$, so $h2 = 9$, so $h4 = 1$. On the other hand, if $b3 = 3$ then $i3 = 1$ so $i4 = 9$ so $h4 = 1$. In other words, it doesn't matter which value we assume that $b3$ takes; either assumption leads to the conclusion that $h4 = 1$, so we can go ahead and assign 1 to cell $h4$.

10 Guessing

The methods above will solve almost every sudoku puzzle that you will find in newspapers, and in fact, you will probably hardly ever need to use anything as complex as multi-coloring to solve such puzzles. But there do exist puzzles that do have a unique solution, but cannot be solved using all the methods above.

One method that will always work, although from time to time it needs to be applied recursively, is simply making a guess and examining the consequences of the guess. In a situation that seems impossible, choose a square that has more than one possible candidate, remember the situation, make a guess at the value for that square and solve the resulting puzzle. If you can solve it, great—you're done. If that puzzle cannot be solved, then the guess you made must be incorrect, it can be eliminated as a candidate for that square, and you can return to the saved puzzle and try to solve it with one candidate eliminated.

Obviously, when you try to solve the puzzle after having made a guess, you may arrive at another situation where another guess is required, in which case a second level of guess must be made, and so on. But since the method always eventually eliminates candidates, you must arrive at the solution, if there is one. In

	1	2	3	4	5	6	7	8	9
a	1 2 5 6	1 2	1 2 5 6 8	1 3 6 8	7	1 2 3 6	9	4	1 2 3 6
b	1 2 6	7	1 2 4 6 8	1 3 6 8	9	1 2 3 4 6	2 3 6 8	1 3 6	5
c	3	1 2 4 9	1 2 4 6 8 9	1 6 8	1 2 4	5	2 6 8	7	1 2 6
d	2 5 9	8	7	4	2 3 5	2 3 6 9	1	3 5 6 9	2 3 6 9
e	4	6	3	1 5 9	8	1 2 9	2 5 7	5 9	2 7 9
f	1 2 5 9	1 2 9	1 2 5 9	1 3 5 6 9	1 2 3 5	7	2 3 4 5 6	8	2 3 4 6 9
g	8	1 2 3 4 9	1 2 4 6 9	7	1 3 4 5	1 3 4 9	3 4 5 6	1 3 5 6 9	1 3 4 6 9
h	7	1 3 4 9	1 4 6 9	1 3 5 9	1 3 4 5	1 3 4 9	3 4 5 6	2	8
i	1 9	5	1 4 9	2	6	8	3 4 7	1 3 9	1 3 4 7 9

Figure 21: A Very Hard Sudoku Puzzle

computer science, this technique is known as a recursive search. Figure 21 is an example of such a puzzle that cannot be cracked with any of the methods discussed so far except for guessing. The solution to this puzzle can be found in section 18.

11 Equivalent Puzzles

There is no reason that the numbers 1 through 9 need to be used for a sudoku problem. We never do any arithmetic with them: they simply represent 9 different symbols and solving the puzzle consists of trying to place these symbols in a grid subject to various constraints.

In fact, the construction of a valid completed sudoku grid is equivalent to a graph-theoretic coloring problem in the following sense. Imagine that every one of the 81 squares is a vertex in a graph, and there is an edge connecting every pair of vertices that lie in the same row, same column, or same block. Each vertex will be connected to 20 other vertices, so the sudoku graph will consist of $81 \cdot 20/2 = 810$ edges. Finding a valid sudoku grid amounts to finding a way to color the vertices of the graph with nine different colors such that no two adjacent vertices share the same color.

Since the symbols don't matter, we could use the letters A through I or any other set of nine distinct symbols to represent what is essentially the same sudoku puzzle. If we take a valid grid and exchange the numbers 1 and 2, this is also essentially the same puzzle. In fact, any permutation of the values 1 through 9 will also yield an equivalent puzzle, so there are $9! = 362880$ versions of every puzzle available simply by rearranging the digits.

If you were trying to calculate how many grids there are, a good approach would be to assume that the top row consists of the numbers 1 through 9 in order, to count the number of grids of that type there are, and then to multiply that result by $9! = 362880$.

	1	2	3	4	5	6	7	8	9
a	5	4	3	4 2 6 8	1 4 6 8	7	1 2 4 8	9	4 8
b	4 7 8	9	1	3	4 5 8	4 2 5	6	4 2 5	4 5 7 8
c	2	6	4 7 8	4 8 9	1 4 5 8 9	1 4 5 9	1 3 4 5 8	4 5 3	4 5 7 8
d	4 3 7 9	8	4 2 5 7	4 6 7 9	4 3 7 6 9	4 6 9	2 3 4 5	2 3 4 5 6	1
e	1 4 3 7 9	1 3 4 7	4 7	5	2	8	4 3	4 3 6	4 6 9
f	6	1 4 5 3	4 2 5	4 9	1 4 3 9	1 4 9	2 3 4 5 8	7	4 5 8 9
g	4 3 7 8	4 5 3 7	4 5 6 7 8	4 6 7 8 9	4 5 6 7 8 9	4 5 6 9	4 5	1	2
h	1 4	1 4 5	9	4 2 6	4 5 6	3	7	8	4 5 6
i	4 7 8	2	4 5 6 7 8	1	4 5 6 7 8	4 5 6	9	4 5 6	3

	1	2	3	4	5	6	7	8	9
a	5	6	1	2 3 6 8	4 2 3 6	7	4 2 6 8	9	2 6
b	2 6 7	9	4	1	2 5 6	5 6 8	3	5 6 8	2 5 6 7
c	8	3	2 6 7	2 6 9	2 4 5 6 9	4 5 6 9	1 2 4 5 6	1 5 6	2 5 6 7
d	1 6 7 9	2	5 6 7 8	3 6 7 9	1 3 7 6 9	3 6 9	1 5 6 8	1 3 5 6 8	4
e	1 4 6 7 9	1 4 6 7	6 7	5	8	2	1 6	1 3 6	3 6 9
f	3	1 4 5 6	5 6 8	6 9	1 4 6 9	4 6 9	1 2 5 6 8	7	2 5 6 9
g	1 2 6 7	1 5 6 7	2 3 5 6 7	2 3 6 7 9	2 3 5 6 9 7	3 5 6 9	5 6	4	8
h	4 6	4 5 6	9	3 6 8	3 5 6	1	7	2	3 5 6
i	2 6 7	8	2 3 5 6 7	4	2 3 5 6 7	3 5 6	9	3 5 6	1

	1	2	3	4	5	6	7	8	9
a	1 2 3 8 9	1 9	1 2 3 9	1 7 8	1 3 7	1 3	4	5	6
b	1 3 8 9	1 4 5 9	6	2	1 4 5 3 7	1 4 5 3	1 7 8 9	1 3 7 8 9	1 7 8
c	7	1 4 5	1 2 3 4	1 4 6 8	9	1 3 4 5 6	1 8	1 2 3 8	1 2 8
d	1 6 9	2	8	1 4 6 9	1 4 5	7	3	1 4 6	1 4 5
e	4	3	1	1 6	8	1 2 5 6	1 7 5 6	1 2 7 6	9
f	5	1 6 9	7	1 4 6 9	1 4 3	1 2 3 4 6	1 6 8	1 2 4 6 8	1 2 4 8
g	1 6 9	1 4 6 7 9	5	3	1 4 7	8	2	1 4 6 7 9	1 4 7
h	1 2	8	1 2 4	1 4 7	6	9	1 5 7	1 4 7	3
i	1 3 6 9	1 4 6 7 9	1 3 4 9	5	2	1 4	1 6 7 8 9	1 4 6 7 8 9	1 4 7 8

Figure 22: Essentially Equivalent Puzzles

In addition to simply rearranging the numbers, there are other things you could do to a puzzle that would effectively leave it the same. For example, you could exchange any two columns (or rows) of numbers, as long as the columns (or rows) pass through the same blocks. You can exchange any column (or row) of blocks with another column (or row) of blocks. Finally, you can rotate the entries in a grid by any number of quarter-turns, or you could mirror the grid across a diagonal.

Figure 22 shows some examples. If the puzzle on the left is the original one, the one in the center shows what is obtained with a trivial rearrangement of the digits 1 through 9 (every 3 in the original was replaced by a 1, every 1 by a 4, and so on). The version on the right is also equivalent, but it is very difficult to see how it is related to the puzzle on the left.

One obvious mathematical question is then, how many equivalent puzzles are there of each sudoku grid in the sense above?

Another interesting mathematical question arises, and that is the following: given two puzzles that are equivalent in the sense above, and given a sequence of steps toward the solution of one that are selected from among those explained in earlier chapters, will those same steps work to solve the other puzzle. In other words, if there is a swordfish position in one, will we arrive at a different swordfish in the other? The answer is yes, but how would you go about proving it?

Notice that the puzzle on the left (and in the center) in figure 22 is symmetric in the sense that if you mark the squares where clues appear, they remain the same if the puzzle is rotated by 180 degrees about the center. Other versions of symmetric puzzles could be obtained by mirroring the clue squares horizontally or vertically. Most published puzzles have this form. This doesn't necessarily make them easier or harder, but it makes them look aesthetically better, in the same way that most crossword puzzles published in the United States are also symmetric.

Another interesting question is this: given a symmetric puzzle, how many equivalent versions of it are there?

12 Counting Sudoku Grids

A sudoku grid is a special case of a 9×9 latin square[3] with the additional constraint barring duplicates in the blocks. There are a lot of 9×9 latin squares:

$$5524751496156892842531225600.$$

Bertram Felgenhauer has counted the number of unique sudoku grids using a computer, and his result has been verified by a number of other people, and that number turns out to be much smaller, but also huge:

$$6670903752021072936960 = 2^{20}3^{8}5^{1}7^{1}27704267971^{1}.$$

If we divide the number above by $9!$ we obtain:

$$18383222420692992 = 2^{13}3^{4}27704267971^{1}.$$

13 Magic Sudoku Grids

	1	2	3	4	5	6	7	8	9
a					4	7			2
b			9				8		
c				1					6
d	6	5	2					7	8
e			8	9					
f				7	8			2	3
g						8			1
h	2	9	4				6		
i	8								

Figure 23: Possible Minimum Sudoku Puzzle

A latin square has all the digits in each row and column. A "magic square" is a latin square where each diagonal also contains all the digits. Is there such a thing as a magic sudoku grid? The answer is yes, and in fact there are a lot of them: 4752, in fact, if we assume that the main diagonal contains the digits in a fixed order. All 4752 of the grids can be completed, and all of them in multiple ways. The puzzle presented in figure 23 is a standard sudoku puzzle, except that it is easier since it requires that each of the major diagonals contains all the digits from 1 to 9.

14 Minimal Sudoku Puzzles

How many locations must be filled with numbers in an otherwise empty grid that will guarantee that there is a unique solution. As of the time this paper was written, the answer to that question is still unknown,

[3] A latin square is a grid where the only constraint is that there be no duplicate entries in any row or any column.

but examples exist of puzzles that have only 17 locations filled that do have a unique solution. Figure 14 shows such a puzzle on the left. Although this puzzle contains the minimum amount of information in terms of initial clues, it is not, in fact, a difficult puzzle. The puzzle to the right in the same figure contains 18 clues, and is symmetric. This is the smallest known size for a symmetric sudoku puzzle.

	1	2	3	4	5	6	7	8	9
a						6	1		
b					9	1	6		8
c	7								
d									2
e	3	7							
f						4			6
g				7	3				
h		8	1						
i				5					

	1	2	3	4	5	6	7	8	9
a			8					3	2
b					6	1			
c			5						
d	6					3			
e	1								7
f				2					8
g							6		
h				8	2				
i	5	3					9		

Figure 24: Minimal Puzzles

15 Measuring Sudoku Puzzle Difficulty

The difficulty of a sudoku puzzle has very little to do with the number of clues given initially. Usually, the difficulty ratings are given to indicate how hard it would be for a human to solve the puzzle. A computer program to solve sudoku puzzles is almost trivial to write: it merely needs to check if the current situation is solved, and if not, make a guess in one of the squares that is not yet filled, remembering the situation before the guess. If that guess leads to a solution, great; otherwise, restore the grid to the state before the guess was made and make another guess.

The problem with the guessing scheme is that the stack of guesses may get to be twenty or thirty deep and it is impossible for a human to keep track of this, but trivial for a computer. A much more typical method to evaluate the difficulty of a puzzle is relative to the sorts of solution techniques that were presented in the earlier sections of this article.

In this article, the techniques were introduced in an order that roughly corresponds to their difficulty for a human. Any human can look at a row, column or block and see if there's just one missing number and if so, figure it out, et cetera.

So to test the difficulty of a problem, a reasonable method might be this. Try, in order of increasing difficulty, the various techniques presented in this article. As soon as one succeeds, make that move, and return to the beginning of the list of techniques. As the solution proceeds, keep track of the number of times each technique was used. At the end, you'll have a list of counts, and the more times difficult techniques (like swordfish, coloring, or multi-coloring) were used, the more difficult the puzzle was.

The rankings seen in newspapers generally require that the first couple of rankings (say beginning and intermediate) don't use any technique other than those that yield a value to assign to a square on each move. In other words, they require only obvious candidates, naked and hidden singles to solve.

Published puzzles almost never require guessing and backtracking, but the methods used to assign a

degree of difficulty vary from puzzle-maker to puzzle-maker.

16 Internet Resources

At the time of writing this article, the following are good resources for sudoku on the internet:

- `http://www.geometer.org/puzzles`: You can download the source code for the author's program that solves sudoku puzzles and can generate the graphics used in this article.
- `http://www.websudoku.com/`: This page generates sudoku games of varying degrees of difficulty and allows you to solve the problem online.
- `http://angusj.com/sudoku/`: From this page you can download a program that runs under Windows that will help you construct and solve sudoku problems. In addition, the page points to a step-by-step guide for solving sudoku, similar to what appears in this document.
- `http://www.simes.clara.co.uk/programs/sudoku.htm`: This site points to some nice descriptions of solution techniques, most of which are discussed in this article.
- `http://www.setbb.com/phpbb/index.php`: This page is a forum for people who want to solve and construct sudoku puzzles as well as for people who want to write computer programs to solve sudoku automatically.
- `http://www.madoverlord.com/projects/sudoku.t`: A downloadable program for the Mac, Windows and Linux that will solve almost any puzzle using logic alone. The distribution comes with great documentation as well, that describes many of the techniques presented here and others besides.

17 Sample Puzzles

This section contains a set of puzzles that require the use of specific techniques to solve them. So if you want to practice with the coloring technique, choose the coloring puzzle, et cetera. Solutions to all of these appear in section 18.

18 Solutions

Figure 27 is a solution to the introductory puzzle in figure 1 on the left and on the right is the solution the the extremely difficult puzzle in figure 21. The other figures are solutions to problems in section 17.

	1	2	3	4	5	6	7	8	9
a	3			2			6		5
b					3				
c	9	8				6			
d			2				8	4	
e	8	7						9	6
f		3	4				2		
g				7				1	2
h					6				
i	7		8			1			9

	1	2	3	4	5	6	7	8	9
a	3	4		2					
b			6	8	3			7	1
c		8				6	3		4
d	1						8		
e				4	2	3			
f			4						7
g	5		3	7				1	
h	4	1			6	2	7		
i						1		6	9

	1	2	3	4	5	6	7	8	9
a	3	6	7						2
b	9		2	3	7				
c	8			2					
d			8	6					
e	5		6		9		2		1
f						5	3		
g						2			5
h					3	4	9		6
i	2						1	4	3

Figure 25: X-Wing (left), Swordfish (center), XY-Wing (right)

	1	2	3	4	5	6	7	8	9
a				8		6		4	
b	5			7			3	1	
c		4	3				8		
d			5						1
e	9			2		1			6
f	6						9		
g			2				6	3	
h		6	9			2			8
i		5		1		7			

	1	2	3	4	5	6	7	8	9
a			7	6			2		9
b		5				4	8		
c			4					1	
d	5			9	4		1	3	
e			2		1		9		
f		8	9		3	2			4
g		6					5		
h			5	8				2	
i	7		8			5	4		

	1	2	3	4	5	6	7	8	9
a	4					8			
b			7	5	3				8
c		9			6		4	1	3
d	5	3					2		7
e									
f	7		6					8	1
g	9	5	4		1			3	
h	3				7	5	1		
i				9					5

Figure 26: Coloring (left), Multi-coloring (center), Hidden Triple (right)

	1	2	3	4	5	6	7	8	9
a	6	2	4	8	7	1	9	5	3
b	1	9	3	4	6	5	8	7	2
c	7	5	8	3	9	2	6	1	4
d	2	1	9	6	4	3	5	8	7
e	5	8	6	7	2	9	3	4	1
f	4	3	7	1	5	8	2	6	9
g	3	4	5	2	1	6	7	9	8
h	8	6	1	9	3	7	4	2	5
i	9	7	2	5	8	4	1	3	6

	1	2	3	4	5	6	7	8	9
a	2	1	5	8	7	6	9	4	3
b	6	7	8	3	9	4	2	1	5
c	3	4	9	1	2	5	8	7	6
d	5	8	7	4	3	2	1	6	9
e	4	6	3	9	8	1	7	5	2
f	1	9	2	6	5	7	3	8	4
g	8	2	6	7	4	3	5	9	1
h	7	3	4	5	1	9	6	2	8
i	9	5	1	2	6	8	4	3	7

Figure 27: Solution to: A Very Hard Sudoku Puzzle

	1	2	3	4	5	6	7	8	9
a	3	4	1	2	9	7	6	8	5
b	2	5	6	8	3	4	9	7	1
c	9	8	7	1	5	6	3	2	4
d	1	9	2	6	7	5	8	4	3
e	8	7	5	4	2	3	1	9	6
f	6	3	4	9	1	8	2	5	7
g	5	6	3	7	8	9	4	1	2
h	4	1	9	5	6	2	7	3	8
i	7	2	8	3	4	1	5	6	9

	1	2	3	4	5	6	7	8	9
a	3	4	1	2	9	7	6	8	5
b	2	5	6	8	3	4	9	7	1
c	9	8	7	1	5	6	3	2	4
d	1	9	2	6	7	5	8	4	3
e	8	7	5	4	2	3	1	9	6
f	6	3	4	9	1	8	2	5	7
g	5	6	3	7	8	9	4	1	2
h	4	1	9	5	6	2	7	3	8
i	7	2	8	3	4	1	5	6	9

	1	2	3	4	5	6	7	8	9
a	3	6	7	1	4	8	5	9	2
b	9	5	2	3	7	6	4	1	8
c	8	1	4	2	5	9	6	3	7
d	4	3	8	6	2	1	7	5	9
e	5	7	6	4	9	3	2	8	1
f	1	2	9	7	8	5	3	6	4
g	6	4	3	9	1	2	8	7	5
h	7	8	1	5	3	4	9	2	6
i	2	9	5	8	6	7	1	4	3

Figure 28: X-Wing (left), Swordfish (center), XY-Wing (right)

	1	2	3	4	5	6	7	8	9
a	1	9	7	8	3	6	5	4	2
b	5	8	6	7	2	4	3	1	9
c	2	4	3	5	1	9	8	6	7
d	8	7	5	6	9	3	4	2	1
e	9	3	4	2	8	1	7	5	6
f	6	2	1	4	7	5	9	8	3
g	7	1	2	9	4	8	6	3	5
h	4	6	9	3	5	2	1	7	8
i	3	5	8	1	6	7	2	9	4

	1	2	3	4	5	6	7	8	9
a	8	1	7	6	5	3	2	4	9
b	2	5	3	1	9	4	8	7	6
c	6	9	4	2	8	7	3	1	5
d	5	7	6	9	4	8	1	3	2
e	4	3	2	7	1	6	9	5	8
f	1	8	9	5	3	2	7	6	4
g	3	6	1	4	2	9	5	8	7
h	9	4	5	8	7	1	6	2	3
i	7	2	8	3	6	5	4	9	1

	1	2	3	4	5	6	7	8	9
a	4	2	3	1	9	8	5	7	6
b	1	6	7	5	3	4	9	2	8
c	8	9	5	7	6	2	4	1	3
d	5	3	8	6	4	1	2	9	7
e	2	1	9	3	8	7	6	5	4
f	7	4	6	2	5	9	3	8	1
g	9	5	4	8	1	6	7	3	2
h	3	8	2	4	7	5	1	6	9
i	6	7	1	9	2	3	8	4	5

Figure 29: Coloring (left), Multi-coloring (center), Hidden Triple (right)